GROUNDED THEORY REVIEW
An international journal

Solutioning

Maria de Hoyos Guajardo, Ph.D. Candidate, M.Sc., B.Eng.

November 2004

Grounded Theory Review, Vol 4 (Issue #1), 59-86

SOLUTIONING

By Maria de Hoyos Guajardo, Ph.D. Candidate, M.Sc., B.Eng.

ABSTRACT

The theory that is presented below aims to conceptualise how a group of undergraduate students tackle non-routine mathematical problems during a problem-solving course. The aim of the course is to allow students to experience mathematics as a creative process and to reflect on their own experience. During the course, students are required to produce a written 'rubric' of their work, i.e., to document their thoughts as they occur as well as their emotions during the process. These 'rubrics' were used as the main source of data.

Students' problem-solving processes can be explained as a three-stage process that has been called 'solutioning'. This process is presented in the six sections below. The first three refer to a common area of concern that can be called 'generating knowledge'. In this way, generating knowledge also includes issues related to 'key ideas' and 'gaining understanding'. The third and the fourth sections refer to 'generating' and 'validating a solution', respectively. Finally, once solutions are generated and validated, students usually try to improve them further before presenting them as final results. Thus, the last section deals with 'improving a solution'. Although not all students go through all of the stages, it may be said that 'solutioning' considers students' main concerns as they tackle non-routine mathematical problems.

GENERATING KNOWLEDGE

An important activity in students' problem-solving process is to generate knowledge about the situation; i.e., to generate relevant data and information and to gain understanding. This is usually conducted at the start of the process, particularly if students know little or nothing about the situation. For this reason, generating knowledge and understanding seems a good place to start the discussion on students' problem solving processes. However, it must be made clear that the need to generate knowledge will continue to emerge throughout the process and that students respond to this need in ways that will be discussed in this section.

A common strategy that students use as they try to generate information and understanding is to reduce the complexity of the situation that they are dealing with. By reducing complexity, students "start at the beginning" and focus on intentionally simplified or even trivial versions of the situation. Students' aim behind reducing complexity is to start gathering the information and understanding that will allow them to eventually move on to more sophisticated cases. Reducing complexity may help students gain access to complex situations by reducing them to simpler, more manageable ones.

> *Numbers which can be expressed as a single prime to a power may be a good place to start…*(Oscar, Liouville, p. 2)
>
> *Right, let's think about this. Start simple and work my way up…*(Hillary, Steps, p. 1)

Students generate information and gain understanding about the situation in many ways. Thus, it is hypothesised that the only limit for students as they try to generate useful information and understanding might be the one imposed by their own creativity and mathematical abilities. The following is a brief list of the types of activities that students conduct for this purpose. The list is not extensive and other activities may be included from further research:

- A common way in which students generate information and understanding is by 'specialising', i.e., by looking at particular aspects of the situation. When students specialise, they focus on isolated aspects of the situation and thus on simplified versions of the problem. For this reason, it may be said that specialising is intrinsically about reducing complexity. Most students specialise at one point or another in their processes and the choice seems to be made in a 'natural' way ("My instinct to this problem is to start from the easiest case.") However, during the course, students were specifically introduced to Mason's (1982) idea of specialising. This fact may account for the students' tendency to specialise and to label their activity in that way.

> *I will start by specialising and using squares, since they seem more straightforward, and then progress to rectangles.* (Hannah, Cartesian Chase, p. 2)

- In order to start making sense of the situation, students sometimes 'import' ideas or information from sources other than the problem and the situation that it presents. These ideas may be relevant to the problem and in the sense that they may help students to better understand the situation and deal with it. Recalling past knowledge or experience are common ways of importing information.

> *I know a similar problem. Diagonals of a Rectangle, which seems to be related and I think I can use my solution. (Emilio, Visible Points, p. 1)*
>
> *Fault line – brings to mind brick walls. In a brick wall you couldn't have such a line because the wall would be too weak. Conjecture that brick*

laying pattern may prove the answer. I will carry on specialising and will come back to this conjecture later. (Kirk, Faulty Rectangles, pp. 1–2)

Students may also import information from other sources such as their notes (or any bibliographical reference, from that matter). Sharing ideas with fellow classmates may also be a way of gaining information and/or understanding. Importing requires borrowed ideas to be evaluated in terms of their relevance and applicability to the present situation. Importing can provide useful information but also presents the risk of considering irrelevant ideas that may have to be abandoned at a later time.

- Another way of generating knowledge is by taking a 'hands-on' approach and carrying out the basic operations that are relevant to the situation. For instance, in 'Faulty Rectangles' students physically constructed rectangles with pieces of domino and observed what combinations could lead to fault-free rectangles. Another example can be given in relation to 'Ins and Outs', where students conducted hands-on investigations by folding pieces of paper and observed the sequences of folds that were generated. Hands-on investigations provide students with first hand experience of the situation and may lead to gaining important knowledge and understanding.

Shall I try playing it? Use a chessboard and a pawn. (Jules, Cartesian Chase, p. 1)

- A way of generating useful information and possibly understanding is by organising the data that is available. This may involve arranging available information in a convenient way so that further information becomes more evident and easier to spot. Tables of values are a common example of organising the data, but any other method for visualising the situation can also be of help.

I will use a table to search for some patterns:

(Keith, Sums of Diagonals, p. 2)

P	1	2	3	4	5	6	7
Dp(1,1)	1	4	10	20	35	56	58

If I try to draw a diagram of the possible outcomes this may help give me a better idea of what is happening and may lead to further development. (Lila, Steps, p. 1)

- An important way of finding more about the situation is by carefully analysing the information that is available or that has been made available. In some cases, information and understanding may emerge easily by looking at the data. In other cases, however, students have to make conscientious efforts in order to generate knowledge. By insistently considering (or reconsidering) available information and trying to understand it, it may be possible to derive further information and understanding from it. This may involve reviewing the data and making deliberate efforts at drawing out observations and ideas.

> *Ok, let's look at our previous example.*
>
> *N=4*
>
> *Stage 1: 1, 2, 4 [Divisors of N]*
>
> *Stage 2: (1), (1, 2), (1,2,4) [Divisors of divisors of N]*
>
> *Is there any significance about the numbers at stage 1?* (Jared, Liouville, p. 6)
>
> *Can't see anything from 3 folds. Only – I guess that the sequence that happened in the previous fold would happen in the current fold again, so 4 folds should start with in in out in in out out, and something else. I want to guess more detail about the 4 folds because I want to prove my prediction is correct. But this is what I can see now.* (Patrick, Ins and Outs, p. 1)

It is not uncommon for students to combine these activities by either conducting them at the same time or by sharing information from one activity to another. For instance, students may take a hands-on approach as they gather information for a table of values. Another example is when students conduct a close analysis of information that has been generated after a period of specialising. As said, there is no imposed limit to what students can do in order to generate information and understanding.

The need to generate knowledge will continue to emerge throughout the process. New information and understanding may be required at any stage, from situations in which students are looking for new ideas to situations where they are trying to take an idea further. In other words, students may incur in the activities discussed above at any time during their process.

Finally, students make reference to the information they observe in the form of written or verbal observations. Trying to gain knowledge about the situation leads students not only to noticing but also to 'making a note' on those new pieces of information that may be relevant in terms of generating a solution. The next subsection looks at the observations that students make as a result of dealing with the data.

Making Observations

The information and understanding that students generate may become manifest in the form of observations. Observations are facts or ideas about the situation that students may find interesting or relevant, and that they choose to point out in a written or verbal way. In some cases, these observations may lead directly to an initial solution.

> *AHA! The pattern behind the centre is just a pattern of the previous one, while those behind is just the opposite way around […]*
>
> *Therefore, if we repeat this, we would be able to generate a sequence after 10 folds. (Karina, Ins and Outs, pp. 1–2)*

In other cases, however, observations may involve information that may or may not be used at a later time.

This is to say that not all observations will be useful in the same way. Some may inform students about ways to generate a solution (like in the example above) and some may provide less central (though not necessarily unimportant) information. In some cases, important observations are easily identified as such. In other cases, it may take the student time and effort to be able to tell whether a certain piece of information is relevant or not.

> *Slope=(4-1)/(4-1)=1.*
>
> *AHA! The gradient of slope 1 is 1. I can use the same method and apply it to slope 2.*
>
> *Slope=(9-1)/(5-1)=2.*
>
> *Aha! I got it! (Patrick, Sums of Diagonals, p. 2)*

> *Obviously, I can only pull out the numbers 1 and 2 and the difference between these is 1.*
> *Hmm… could this always be the case (wild guess)? Or is it too early to tell.* (Aminta, Hat Numbers, p. 1)

When students come across an observation, sometimes they adopt what can be called a 'pragmatic' approach. Adopting a pragmatic approach involves focusing not only on the observation itself but also on how it can be used for generating a solution. When students adopt a pragmatic approach towards making observations they ask themselves questions like "How can this [idea, fact, etc.] be used?" A pragmatic approach can help students decide more efficiently whether an idea is useful and how.

The examples below (as well as Patrick's example above) illustrate cases where students considered observations in a pragmatic way. As the second example below suggests, a pragmatic approach may help students discriminate unimportant ideas and thus may help in making their process more efficient. Thinking in terms of how ideas can be used seems to lead to starting to generate a solution sooner than if observations are made without considering their usefulness or applicability.

> *The answers for 2 and 5 give the answers for 10. Does this work for other numbers?* (Julia, Liouville, p. 5)
>
> *Points (i, j), where i, j are positive.*
>
> *Defined to be BELOW (m, n) where m, n are positive when ≤m and ≤n.*
>
> *∴ (i, j) is below itself – not particularly important. (Dylan, Visible Points, p. 1)*

Key Ideas

Having discussed how students generate knowledge about the situation and how this knowledge becomes manifest, this section will look at 'key ideas' as knowledge that is crucial to solving the problem and that students employ directly to generate a solution. The first subsection discusses 'looking for patterns' as ways of looking for *key* ideas by investigating the situation in a particular way. The second sub-section discusses 'key searching' as a way of looking for key ideas in a more direct way.

As said in the previous section, some of the observations that students make during problem solving lead directly to generating a solution. Since these observations usually refer to crucial aspects of the situation they can be called key ideas. Students usually base their solutions on a key plan or idea that provide hints as to how a solution can be obtained. In order to deal with 'Diagonals of a Rectangle', for instance, students used the fact that there is a relationship between the highest common factor of the rectangle's dimensions and the number of rectangles crossed. This fact was the key idea on which most (if not all) students who provided a solution for this problem based their processes.

Key ideas sometimes emerge as sudden realisations of important aspects of the situation. These ideas may appear as important breakthroughs (as the student below suggests) and give students the feeling of having discovered how to generate a solution.

> *AHA! This is a huge breakthrough! Anything that happens before the row marked (*) is not important. As long as we can guarantee that our opponent moves to (*), we have won, since we can then move to a definite win position.* (Leonard, Cartesian Chase, p. 5)

In other cases, key ideas emerge as less of a surprise. In these cases, key ideas may come gradually as knowledge and understanding increase.

In either case, it seems that being able to arrive at a key idea requires a good deal of understanding of the situation. When students are able to see a key idea, they are also able to see its significance, its importance in relation to the situation and how it can be of use. In relation to this, Raman (2003) observed that the key ideas that more experienced solvers use to provide a mathematical proof "give a sense of understanding and conviction" and show "why a particular claim is true" (p. 5). In more general terms, Barnes (2000) suggested that when students and more experienced mathematicians are able to see a key idea the following takes place:

> *…there is a claim to a sudden realisation of new knowledge or understanding. Usually this knowledge is 'seen' with great clarity, or experienced with a high degree of confidence or certainty.* (Barnes, 2000, p. 34)

Key ideas can be seen as the product of gathering sufficient relevant knowledge and understanding to be able to start generating a solution. The following sub-sections look at ways in which students generate and search for key ideas.

Looking for patterns

Looking for patterns can be considered as a way of learning about the situation that can lead to finding key ideas. When students look for patterns, they are usually looking for particular features of the situation can lead them to start generating a solution. Students look for patterns hoping that, when they find one, they will be able to transform it into a formula or to make a general statement about the situation.

> *I shall look for patterns which might lead me to a formula of some kind.* (Lila, Sums of Diagonals, p. 1)

Looking for patterns can be a useful activity that generates relevant information. For instance, noticing a pattern in the way the creases were formed in the 'Ins and Outs' problem allowed students to tell how the creases for the 10th fold would look like. Furthermore, as students look for patterns, they may also gain understanding and learn about the situation. Thus, in many cases, looking for patterns can be a fruitful activity.

However, looking for patterns can also become a 'blinding' activity that prevents students from gaining the necessary information and understanding. When students focus mainly on looking for patterns and neglect trying to see other aspects of the situation, the possibility of gaining useful information seems to decrease. In the example mentioned above, most students were able to see how creases were formed and thus were able to tell how the 10th fold would look like. However, very few students were able to provide a general (non-recursive) formula for this sequence. Students that were able to provide a general formula did so not by looking for patterns but by gaining a deeper understanding of how the sequence of 'ins' and 'outs' was generated. In contrast, students that focused mainly on looking for patterns (as illustrated below) were able to provide a recursive formula but failed to provide a general one.

> *I can't see a pattern or anything jumping at me.*

But by counting the number of 'ins' and 'outs' in any number of folds I can see that each one seems to be an odd number.

E.g.,

> *Just comparing the difference between the number of 'ins' and 'outs' seems to show that they are powers of 2.* (Rita, Ins and Outs, p. 2)

Number of folds	**Number of I's and O's**
1	1
2	3
3	7
4	15
5	31

Thus, it may be said that looking for patterns can provide some very useful information. In order to provide a more satisfactory solution, however, further information and understanding need to be generated as well. Focusing on trying to find particular information about the situation can lead to a dead end as it prevents students from genuinely learning about the situation. 'Key searching', as will be discussed in the next sub-section, is a way of looking for key ideas that is related to this aspect of looking for patterns.

Key Searching

As mentioned above, key ideas allow students to start generating a solution. Finding a key idea is certainly related to successful problem solving, and students seem to be aware of this. For this reason, students may look for key ideas by looking for patterns. Another way of looking for key ideas is by 'key searching'. Key searching means looking for key ideas in a direct way by trying discover special features about the problem or by trying to find "what is so special" about the situation.

> *I'm looking to see if the number left in the hat has some special quality…*
>
> *Still stuck! Maybe I should go back and try the odd numbers. After all, as this may be the missing clue to the solution…*(Aminta, Hat Numbers, pp. 2–4)

As students try to gain knowledge and understanding of the situation, it is very likely that they will eventually come across key ideas. Paradoxically, however, key ideas are less likely to emerge if students focus on actively seeking them. The reason for this may be that searching for key ideas may divert students' attention from trying to learn about the situation. During key searching, students seem to be so concerned about trying to find some "special" clue or quality that they may neglect other important information. In the case of the Liouville problem, for instance, some students spent most of their process trying to figure out what was so special about sequences of numbers that if added and then squared give the same value as when they are cubed and then added. In these extreme cases, students were unable to make any significant progress and were not able to identify any of the key ideas that allowed other students to generate a satisfactory solution.

When students search for key ideas, they may ignore important information that, if not a solution in itself, can be used towards that end. Furthermore, in some cases, students that search for key ideas seem to ponder on the problem rather than on trying to gain a broader understanding of the situation.

In general, not all students incur in key searching and those who do may eventually abandon this activity and try to generate information and understanding. However, the implications of key searching make this activity an important one to consider. There is no evidence to suggest that key-searching is related to mathematical background. What can be suggested is that key-searching may be related to the features of the problems involved. This hypothesis is supported by the fact that more students key-searched in the 'Liouville' problem than in any other. There is not sufficient evidence to state take this hypothesis further. This issue can only be suggested for further research.

GAINING UNDERSTANDING

The above sections deal with the way students generate knowledge during their problem-solving processes. This knowledge constitutes the information and understanding that will allow them to deal with the problem and eventually to achieve a solution. This section deals more closely with the issue of gaining understanding. This issue plays an important role in being able to generate a solution and most students will seek to gain understanding about the situation. However, as it is discussed below, students may also ignore or avoid trying to gain understanding and concentrate on manipulating data.

A good place to start a discussion on the characteristics of gaining understanding during problem solving is by considering the following quote from Thurston:

> *On a more everyday level, it is common for people first starting to grapple with computers to make large-scale computations of things they might have done on a smaller scale by hand. They might print out a table of the first 10,000 primes, only to find that their printout isn't something they really wanted after all. They discover by this kind of experience that what they really want is usually not some collection of answers – what they want is understanding.* (Thurston, 1995, p. 29; emphasis in the original)

Although Thurston's assertion was made in reference to professional mathematicians, it may be said that it applies to many students as well.

Gaining understanding is an important aspect of the problem solving process. Most students try to gain understanding of the situation to be able to start generating a solution. As a student put it, it is easier to generate a solution by "understanding the underlying principles" of the situation. In general, it seems that having a better understanding of the situation empowers students and allows them to generate a solution and take it further.

> *I can't believe how I missed how every entry in the grid is the product of its coordinates…*

> *This means that given any coordinates we can work out what the entry is.* (Nadia, Sums of Diagonals, p. 4b)

An important way of gaining understanding is by reasoning in terms of how the data is created, or how it stems from the situation. Although not all students try to gain understanding in this way, and those who do may not do so all the time, it may be said that thinking in terms of how information is created is a common practice. Thinking in terms of how the sequences of 'ins' and 'outs' were created, for instance, provided students with useful understanding of the situation. In most cases, this allowed them to generate an initial solution for the 'Ins and Outs' problem. The following quotes illustrate the type of reasoning that was conducted in an attempt to gain understanding in relation to this problem.

> *What I'm going to do is take the five folds sequence and identify which creases come from which fold. (Lydia, Ins and Outs, p. 7)*
>
> *Maybe I should start to think about things on a more subtle level. What actually happens every time I add a crease of paper? I'll try to get this into a diagram.* (Leonard, Ins and Outs, p. 4)

When students try to think in terms of how the data is created, they usually gain a kind of understanding that allows them to make informed decisions on what to do next. In other words, they achieve what Skemp (1976) called 'relational understanding'. This type of understanding allows students to know "both what to do and why" (p. 20) and for this reason it is usually an important asset during problem solving. The understanding achieved by the students in the following examples is relational in the sense that it provides information that can be useful for understanding the situation and deciding what to do next. Furthermore, their understanding seems to have been generated by reasoning in terms of how what they observe stems from the observed situation:

> *Let's try to think logically about specifically when a diagonal would pass through a corner.*
>
> *AHA! I think the diagonal will pass through a corner when n and m have a common factor greater than 1. This makes a lot of sense because it implies that the rectangle can be split up into smaller rectangles with the same diagonal, and therefore the diagonal would pass through the corners.* (Hannah, Diagonals of a Rectangle, pp. 3–4)

Finally, considering the benefits of trying to think in terms of how the data is created may look as if all students worked naturally in this way. However, this is not the case. Students with stronger mathematical backgrounds are usually keen on reasoning in terms of how the data stems from the situation. On the other hand, students for whom mathematics is not a main subject seem more prone to look for patterns without considering the situation that gives rise to the data. The reasons for this behaviour are difficult to trace. It can be speculated that thinking in terms of how data relates to the situation requires students to combine thinking about the situation while, at the same time, trying to identify useful patterns. Thus, some students may unconsciously avoid such an increased complexity and choose to focus on only one task at the time. In such situation, they may prefer to work on the simpler one which will be, presumably, trying to spot patterns. This, however, is a tentative explanation; a more grounded explanation certainly requires further research.

GENERATING SOLUTIONS

The previous sections looked at how students generate knowledge about the situation. It was discussed how students make key ideas available and what courses of action may hinder their emergence. Some ways in which students gain understanding about the situation were also considered. In spite of its importance, it may be said that generating knowledge is not the final aim of problem solving but a means of making necessary resources available. The aim of problem solving is to generate a solution and students will start attempting to do this as soon as sufficient knowledge has been gathered. Two ways in which students may try to generate a solution is by reasoning deductively and inductively. Reasoning in terms of how data is generated from the situation can also play an important role in generating a solution.

In order to generate a solution, students may rely on deductive reasoning. In other words, they may follow logical implications from one idea to another until a conclusion is reached. Reasoning deductively seems to be held in high regard by most students since, whenever possible, they will try to arrive at a solution in this way. In the Liouville problem, for instance, most students' first attempt at generating a solution involved providing some version of the following deductive argument.

> *A prime number n has divisors 1 and n only, by definition.*
>
> *1 has one divisor (1)*
>
> *n has two divisors (1, n)*
>
> *The sum of the number of divisors or divisors is therefore 1+2=3 and squared this is 9.*
>
> *The sum of cubes of the number of divisors or divisors is 13+23=9.*
>
> *So the two numbers are equal for prime numbers.* (Julia, Liouville, p. 2)

Also, as one student put it:

> *I generally try to use deduction. Deduction is 'more valid' in mathematics although I often use inductive arguments.* (Leonard, informal interview)

When students reason deductively, they sometimes base their arguments on a relevant piece of mathematical knowledge. This piece of knowledge may consist of mathematical concept or a procedure. In other words, students may build a deductive argument by applying a concept or a definition in an ingenious way or by making use of a familiar mathematical procedure. In the example above, the student based her deduction on the mathematical definition of 'prime number'. The way she made use of this definition allowed her to generate a logical chain of reasoning and to achieve an initial solution. As for applying a mathematical procedure, the Arithmagons problem provides a good example. In most solutions to the 'Arithmagons' problem it was common for students to base their arguments on procedures for solving systems of linear equations. Although making use of procedures may be more straightforward than deciding how to apply a concept, in the sense of constructing logical chains of reasoning, the former can also be considered a deductive argument.

Whenever there is the possibility of generating a deductive argument from the knowledge and information available, students will usually follow this route. When this is not the case, one option is to continue trying to generate information and understanding until it is possible to generate a deductive argument. Another option is to start trying to generate a solution by induction.

Reasoning inductively involves making tentative conjectures or generalisations out of the information that is available. Making deductions involves deriving ideas that are a logical consequence of the information available. In contrast, when students reason inductively, they not only consider the information that is available (and the logical implications of this information) but also draw upon other less factual sources such as previous (possibly informal) knowledge and experience. This knowledge and experience may arrive in the form of insight or intuition, or in the form of 'intuitive guesses', as Fischbein and Grossman (1997) put it. It is the combination of empirical data with other sources of knowledge what usually makes inductive reasoning a fascinating process.

> *All the results are in a range 48–63…*
>
> *Notice that the last two results are equal.*
>
> *Conjecture 1: the percentage of visible points converges to a number.*
>
> *Conjecture 2: the convergent number x=48.7%.* (Aminta, Visible Points, p. 4)

Generating ideas inductively may lead to inaccuracies or even to incorrect solutions. This is not to say that deductive reasoning is foolproof. What this suggests is that, due to the nature of inductive reasoning, students sometimes have to accept, and deal with, the fact that they are working with imperfect results. However, this is usually not a serious problem since ideas can be re-examined and modifications can be made. Moreover, checking whether a tentative solution is correct and makes sense allows students to improve their solution and increases their knowledge and understanding of the situation. This, together with the fact that an initial solution – i.e., a starting point – is already available, seems to outweigh the possible drawbacks of generating a solution in an inductive way.

As said, most students will try to work deductively if at all possible and if not they may choose to work inductively. However, inductive and deductive reasoning are not mutually exclusive as this generalisation may suggest. In fact, it may be said that students combine both approaches and that they complement each other. For instance, after reasoning inductively and generating some feasible conjectures, students may recur to deductive reasoning to show that these are always true.

Besides reasoning inductively and deductively, students may generate a solution as a result of reasoning in terms of how data is created. The previous section discussed how thinking in terms of how data is created may provide students with information as to what to do next and why. Since this information is easily translated into a solution, reasoning in terms of how data is created can be considered as another way of generating a solution that is different to both induction and deduction. Simon (1996) observed a similar situation. He suggested that students may invent or infer situations to explain how data is created and that this may allow them to generate a solution. The following example illustrates the case of inventing a situation to explain how data is created and how the understanding that it provides can be used to generate a solution.

> Ms. Goodhue: *Mary, could you make an isosceles triangle by specifying two angles and the included side?*
>
> Mary pauses and then punches in equal angles.
>
> Ms. Goodhue*: Can you tell me what you did?*
>
> Mary*: Well, I know that if two people walked from the ends from this side at equal angles towards each other, when they meet, they would have walked the same distance.*
>
> Author [Martin Simon]: *What would happen if the person on the left walked at a smaller angle to this side?*

> Mary: (Without hesitation) *Then that person would walk further [than the person on the right] before they meet…* (From Simon, 1996, p. 199)

Thinking in terms of how data is created can be seen as a way of gaining deep understanding of the situation that helps generating a solution. Solutions achieved in this way tend to be more 'transparent' than solutions arrived at by deduction or induction. When students reason in terms of how data is created, it may become evident how a solution should look like and why.

Guessing and Ungrounded Ideas

It was mentioned before that tentative solutions that are generated inductively or in any other way are usually a good place to start generating a more comprehensive solution. However, there does seem to be an exception to this case. In some cases, students' apparently inductive reasoning can be better explained as 'guessing'. When students guess a solution, their reasoning is unclear and it is usually difficult to tell where ideas come from. Yet, from the comments that students make, it usually becomes evident that they may be testing their luck and proposing ideas without going through conscientious reasoning about the situation.

> *Try completely new approach. Convert sequence into a straight number using binary representation (might get lucky). (Sebastian, Ins and Outs, p. 5)*
>
> *We can see by looking at the diagram that there are three points that would not be visible. Could I work this out algebraically so that it applies to any size grid square?*
>
> *Maybe it could be (i–j)/j, that would be (9–3)/3=6/3=2. That doesn't work!*
>
> *Maybe (i–j)/i would be better: (9-3)/2=3. Would this work for other (i, j)? […]*
>
> *There only seems to be two points which means that my formula is not correct.* (Gina, Visible Points, pp. 3–4)

Ideas that are arrived at by guessing are usually ungrounded, i.e., they are more the product of inventiveness than of carefully analysing the data. Although the relation between guessing and ungrounded ideas is somewhat evident, guessing a solution is not the only way in which students may generate this type of ideas. Trying to invent a situation to explain how data is created may also lead to generating ungrounded ideas, particularly when used without considering sufficient empirical data. In other words, in an attempt to provide an account of how data is generated or of how the "system in question works", students may fall into ‘making up’ an explanation that is more the product of their ingenuity than of what they know about the situation.

Ungrounded ideas tend to be inconsistent and thus can lead to problems and frustration. This was the case of a student that provided an interesting explanation as to why it is not possible to build a fault-free rectangle (see the ‘Faulty Bricks’ problem). Since fault-free rectangles can be built, and since the explanation was the result of the student’s creativity, she found it hard to elaborate the argument further. In general, although ungrounded ideas can be problematic, a positive aspect is that the frustration that they cause may become, in some cases, a good place for starting to learn about the situation.

Summarising, students may generate an initial solution by reasoning deductively, inductively or in terms of how data is generated. Although students may have a ‘predilection’ for deductive reasoning, it seems that this predilection is based more on their beliefs about mathematics (deductive reasoning being ‘more valid’) than on the results that they obtain from reasoning in this way. Inductive reasoning may allow students to generate initial solutions that can later be improved. Thinking in terms of how data is generated is a good way of generating ‘transparent’ solutions. Although the last two types of reasoning may not be the students’ first choices, they can be efficient ways of generating results.

Once a solution is generated, it may be validated and/or improved. The next two sections look at ‘validating’ and ‘improving’ results, respectively.

VALIDATING RESULTS

During their problem solving processes, students look for ways of validating the ideas that they are generating. To do this, they may try to validate their results in terms of whether they are correct and make sense. In other words, students try to verify that their results are correct and seek to explain why this is the case. When students validate their results in this way their main concern is being on the ‘right track’ and having a clear understanding of the situation. Thus, the arguments that they produce can be considered as personal ‘proofs’ aimed at convincing themselves, their peers and possibly even a sceptical reader trying to follow their process (i.e., convincing oneself, a friend and an enemy, in Mason’s (1982) terms).

Once students have achieved a satisfactory solution, they sometimes seek to provide a formal mathematical proof of their work. However as the quote below suggests, providing a formal argument seems to have a different purpose than making sure that a solution is correct and makes sense.

This certainly seems to hold for all m, n [where m and n are natural numbers], but whether or not I can prove it is a different matter. (Leonard, Diagonals of a Rectangle, p. 19)

It seems that trying to provide a formal mathematical argument that proves that a solution is true is more a way of improving a solution than of making it convincing for themselves and for others. For this reason, providing a formal proof will be discussed in the next section below ('Improving Results').

Making Sure Results are Correct and Make Sense

Students may validate their results by verifying that their ideas are correct and make sense. In order to verify that results are correct, students may review their reasoning and look for any errors or inconsistencies. For instance, they may check that suitable procedures were chosen and that they were properly conducted. Besides verifying their procedures, students may check to see whether their generalisations work in particular cases. If the results obtained from particular cases are as expected or match with previous data, then they can be accepted. Verifying that results are correct allows students to move on, whereas noticing any inconsistencies will require them to go back and try to correct them.

Now I want to check it again that my result is right before I go any further from here. Therefore I count the number of grid squared that are touched by the diagonal again from the grid squares that I have already drawn. And it's correct. (Anibal, Sums of Diagonals, p. 4)

I will now see if it works for the numbers I have so far. (Jasmine, Sums of Diagonals, p.6)

Check: Does this match the examples I have tried so far? (Julia, Liouville, ca. p. 10)

Students may verify that the ideas being generated make sense by looking for explanations as to why they must be true. Explaining why an idea is true reassures students that the solution that they are generating is congruent with their knowledge and with what they know so far about the situation. Furthermore, when students try to make sure that their generated ideas make sense they may resort to thinking in terms of how data is created. Understanding of how the situation 'works' and how the data is derived from the situation provides students with ideas that can be used to explain why a solution must be true.

> *Why does this work? Aha! Looking at any diagonal, moving down one adds 1 to the first element, 2 to the second, etc. And then finally one more element equal to the new 'x'. (Marcus, Sums of Diagonals, p. 3)*

Trying to verify that results are correct and making sure that they make sense are related activities that are usually combined. In many cases, after checking that their results are correct, students may proceed to explain why this is the case. The following quote illustrates this situation.

> *This looks like the number of creases is 2a-1.*
>
> *Check for a=6.*
>
> *From previous formula creases = 31+32=63=26-1.*
>
> *I can see this would be true because each time I am doing n+(n+1) to get the next term which is equal to 2n+1, so each time I am doubling the previous number (which is less than 2n as 1 is one less than21=2) which would give me 2n=2 and then adding one so I get 2n-1. (Jasmine, Ins and Outs, p. 4)*

This is not to say, however, that verifying that generated ideas are correct implies that students will proceed making sure that they make sense. After all, not all students are able to conclude their process by saying:

> *My calculations do work and make sense, and I think the answer is reasonable.* (Hannah, Faulty Rectangles, p. 11b)

In some cases, students may not be interested in explaining why ideas are true so long as they seem correct. In other cases, students may be able to verify that their results are correct but may find it difficult to provide an explanation as to why this is the case.

> *It does seem to be the case that the Liouville results are always identical, regardless of the chosen starting number. Sadly, I have no theories as to why this occurs.* (Conrad, Liouville, p. 5; emphasis added)

Continuously trying to verify that ideas are correct and make sense ensures that inconsistencies are brought to the fore and provides an opportunity to amend them. In fact, it seems that verifying that ideas are correct and make sense, and making the necessary modifications, plays an important role in successful solutioning. Inglis and Simpson (in press) suggest that it is error-correcting – rather than error-free processes – that may account for the fact that mathematicians perform better than non-mathematicians in logic tasks. Furthermore, in a study of collaborative problem solving in combinatorics, Eizenberg (2003) found that it was not peer collaboration that was directly related to successful problem solving but that successful problem solving is closely related to 'control behaviours', i.e., to constantly monitoring whether ideas are correct and making the necessary modifications. In the author's words:

> *Our study provides evidence that success in problem solving in combinatorics is not a direct outcome of collaborative problem solving. It is mostly a result of enhanced control behavior.* (Eizenberg, 2003, p. 399)

In spite of the benefits of validating results, students do not always stop to verify that their results are correct and make sense. As said, validating solutions in the ways discussed here may help to reassure students that they are on the 'right track' in terms of the ideas that they are generating. This, in turn, will allow them to continue with their solving process or, in other words, to 'move on'. In some cases, however, being able to move on can be more important than whether results are correct and make sense. In such cases, students may simply avoid trying to validate their results or will do it in superficial ways. For instance, they may check that results are true in one or two known cases. In this way, even if results are inaccurate, this will not necessarily prevent them from continuing to work towards a concluding solution.

> *Number of rectangles formed is 3n+(n–4). E.g., when 5 dominoes are used 15+1=16.*
>
> *That seems to work! I will test the formula out when more dominoes are used.[Continues to work with 3x1 rectangles]* (Gina, Faulty Rectangles, p. 3)

Being able to validate a result may provide students with an acceptable solution. However, unless the student had already been working on improving this solution, it is very likely that it will not be final but one that needs to be improved. The next section looks at ways in which students may seek to improve a solution once it has been achieved.

IMPROVING THE RESULTS

This section looks at what can be considered as the last stage of the solutioning process. Once a solution is achieved, students usually acknowledge the need to improve their results. This is particularly true when students feel that their answer is correct but not ready to be presented as it is. If time and mathematical knowledge allow, they may try to improve their results by providing a formal mathematical proof or by extending their results to other domains. Alternatively, they may try to express their solution in more concise ways.

> *OK – I'm happy that's worked out in that case. I'm definite there is a more elegant explanation which might be worth looking for. Argument sounds a little awkward to me at the moment – could do with being more persuasive.*
>
> *Right. Review here – there's a few different ways to go…*
>
> *Have shown for odd x even, if I could show for even x even I'd be done!* (Rafael, Faulty Rectangles, p. 12)

> *I wonder if I could improve this further by rewriting my formula as a closed expression, i.e., an equation in x and n with no summation signs.* (Hillary, Sums of Diagonals, 15)

Improving a solution can be a straightforward task that involves making simple modifications or additions. However, this is not always the case and the work that students need to conduct to improve a solution can vary from being straightforward to very laborious and time-consuming. In most cases, improving a solution will involve dealing with situations that are more complex than when an initial solution was generated. Having to deal with progressively more complex situations can make it difficult – or even impossible – for some students to improve their solutions further. The probability of this being the case seems to be higher when students lack the necessary mathematical background to deal with more sophisticated mathematical ideas. Lack of time or energy can also prevent students from improving their solutions. Under these circumstances, some students will decide to stop their process and will present their solution as it is.

> *Reached a dead end at the moment so I am unable to progress any further. If I had been able to solve this problem properly I could have also extended it to look at the rest of the items on my brainstorm. (Lydia, Cartesian Chase, p.13)*

Students who are able to improve their solutions recognise that it is almost always possible to take them even further. However, they only have to continue improving their solution until a seemingly acceptable solution is found. Such a solution is one that is clearly (and if possible, formally) stated and that accounts for a variety of cases.

Trying to Provide Formal Mathematical Proofs

One way in which students may seek to improve their results is by attempting to produce a formal mathematical proof of their work. Once a satisfactory solution or initial solution is generated, students may try to improve it by providing a more rigorous argument. Providing a formal mathematical argument is a way of putting an already satisfactory solution in such a way that it can be presented as a final product to others. In other words, providing a formal mathematical proof involves elaborating a deductive argument that not only satisfies the student's understanding but also satisfies certain mathematical requisites.

Producing a formal mathematical proof is something that some students do as part of their processes. For instance, in 'Sums of Diagonals' various students proved their general formulas by mathematical induction. However, in general, it may be said that providing a rigorous mathematical proof is usually considered a secondary aim. For some students, the fact that the results are reliable should be evident from the way they were generated and validated.

> *I believe I have the correct answer, although I have no concrete proof. I believe that, as a possible extension, it would be possible to get an answer involving trigonometry…This would be a concrete 'proof' of the answer but it isn't very easy to show. Other extensions [could be]…* (Roberto, Diagonals of a Rectangle, p. 5)
>
> *My formulas are very general and because of the way they were obtained they don't really need any formal proof or justification, as these are evident in the method.* (Nadia, Sums of Diagonals, p. 7b)

In general, students seem more concerned about producing arguments that are convincing, both for themselves and for a sceptical reader than of providing a formal mathematical proof. Moreover, when it comes to improving their solution, they seem to be more concerned about extending their results, as will be discussed next.

Extending

Once students generate a solution, it is not uncommon for them to try to improve it by extending it. Students extend their solutions by showing that they account for all possible cases or by making their results valid for a wider domain.

When students generate a solution, they sometimes notice that the ideas or the methods that they used can be applied to other situations as well. In other words, they notice that some of their ideas can be transferred and thus be made useful for solving, or dealing with, other cases – i.e., for extending.

> *Aha! If I can do this for a number with two divisors that are prime, I could probably do it for a number with exactly 3, 4, … or more non-trivial divisors, all which are prime.* (Jason, Liouville, p. 3)
>
> *Can I use the same process as earlier to generate more even x even fault free rectangles?* (Camille, Faulty Rectangles, pp. 2b–3)

Although transferring means that previously developed ideas will be used in other situations, this is not necessarily a simple task. Transferring may require students to make some changes to the ideas or procedures to be transferred to make them suitable for the new situation. These changes can be relatively simple, such as when students decide to introduce a new, more efficient notation.

> *The largest secret number 'a' was found by adding the two largest side numbers and subtracting the remaining side numbers…I think [this] rule is most likely to work with arithmagons with >3 sides.*
>
> *As I am seeing a general rule for arithmagons with n sides, I will need to alter my notation for improved clarity. Instead of x, y and z for the side numbers I will use s1, s2, s3, …, sn…* (Jules, Arithmagons, p.5)

In other cases, adapting previously used methods or ideas can be complicated or even impractical.

> *My proof that there was a path came from visualising, again, what the path should be, since anything other than the circle seemed unlikely, and bearing in mind the complete symmetry of the circle. Unfortunately, this reliance on the symmetry of the circle meant I couldn't extend the theory to irregular circles very easily.* (Albert, Jogger's Dog, Commentary)

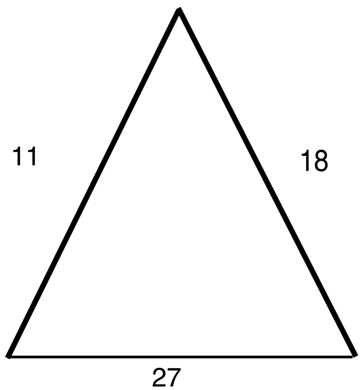
In some cases, adapting can be a considerably complicated activity. In situations like this, students will find that looking for new ways of generating a solution may be a better option. In a way, finding new ways of solutioning may suggest that students will need to start the solving process all over again. However, this is not the case. The knowledge and understanding that students have gained about the situation are very likely to make this 'new' process a more efficient one. Of course, this will be the case only if students persist in extending their solutions. They may well decide to stop their process at this stage.

APPENDIX 1 – THE PROBLEMS (IN ALPHABETICAL ORDER)

Arithmagons:

A secret number is assigned to each vertex of a triangle. On each side of the triangle is written the sum of the secret numbers at its ends. Find a simple rule for revealing the simple numbers.

For example, secret numbers 1, 10, 17 produce:



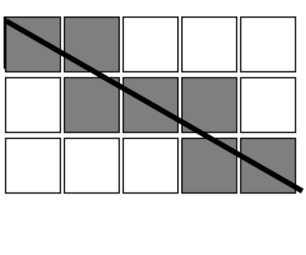
Generalise to other polygons.

Cartesian Chase:

A game of two players is played on a rectangular grid with a fixed number of rows and columns. Play begins in the bottom left hand square when the first player puts a counter. On his turn, a player may move the counter one square up, one square right or one square diagonally (up and right). The winner is the player who gets the counter to the top right square.

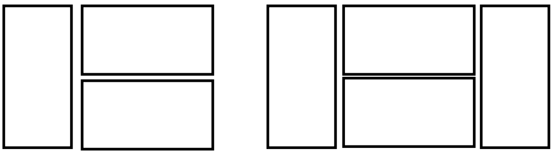
Diagonals of a Rectangle:

On squared paper, draw a rectangle and draw in a diagonal. How many grid squares are touched by the diagonal. E.g.



Faulty Rectangles:

These rectangles are made from ‘dominoes’ (2 by 1 rectangles). Each of these large rectangles has a ‘fault line’ (a straight line joining opposite sides).



What fault free rectangles can be made?

Hat Numbers:

A hat contains 1992 pieces of paper numbered 1 through 1992. A person draws two pieces of paper at random from the hat. The smaller of the two numbers drawn is subtracted from the larger. That difference is written on a new piece of paper which is placed in the hat. The process is repeated until one piece of paper remains. What can you tell about the last piece of paper left?

Ins and Outs:

Take a strip of paper and fold it in half (always placing the right hand edge on top of the left hand edge). Unfold it several times and observe the sequence of 'in' and 'out' creases. For example, three folds produces:

in in out in in out out

What sequence would arise from 10 folds?

Jogger’s Dog:

A jogger runs, at a constant speed, around a circular track. The jogger’s dog runs, always toward the jogger, at constant speed. What sort of paths does the dog describe?

Liouville:

Take any number and find all of its positive divisors. Find the number of divisors of each of these divisors. Add the resulting numbers and square the answer. Compare it with the sum of the cubes of the numbers of divisors of the original divisors

Sums of Diagonals:

Investigate the sums of diagonals of different slopes in the grid below.

1	2	3	4	5	6	7	8	9	…
2	4	6	8	10	12	14	16	18	…
3	6	9	12	15	18	21	24	27	…
4	8	12	16	20	24	28	32	36	…
5	10	15	20	25	30	35	40	45	…
6	12	18	24	30	36	42	48	54	…
7	14	21	28	35	42	49	56	63	…
8	16	24	32	40	48	56	64	72	…
9	18	27	36	45	54	63	72	81	…
…	…	…	…	…	…	…	…	…	

Steps:

You are standing at the beginning of an infinitely long path, as shown below:

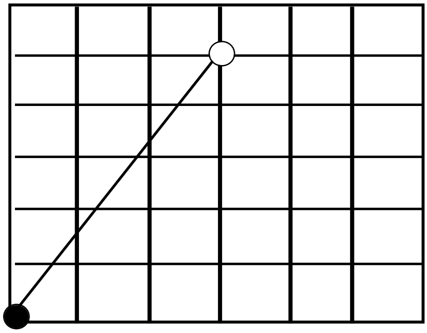
You throw a fair coin which has the number "1" written on one side, and he number "2" on the other. You walk forward the number of steps shown on the side of the coin that lands face up. For example, if you throw the coin and it comes up "2" you take 2 steps forward to land on the 3rd step of the path - 2 steps from where you were on step number 1.

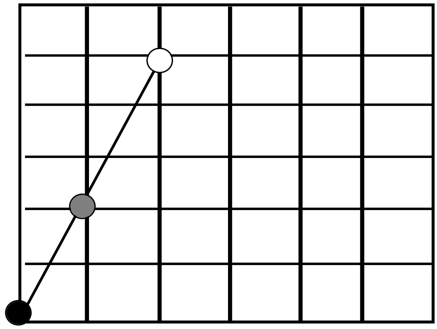
You now repeat the exercise - throw the coin again and walk forward the number of steps that comes up on the coin. If you throw the coin 24 times you are certain to have landed on, or past, spot number 25. What is the probability that you will land on step number 25?

Visible Points:

A point (i, j) in the plane, with non-negative integers coordinates i and j, is below a point (m, n) with non-negative integer coordinates when and i £ m and j £ n.

A point (m, n) in the plane, with m and n non-negative integers, is visible from (0, 0) if the straight line joining (0, 0) to (m, n) passes through no other points below (m, n).





As m and n increase, what percentage of points is visible from (0, 0)?

Author

Maria de Hoyos Guajardo, Ph.D. Candidate, M.Sc., B.Eng.

Research Fellow

Institute for Employment Research

Warwick University

Coventry UK

CV4 7AL

Correspondence

Phone: +44 (0) 779 6614243

E-mail: maria.de-hoyos@warwick.ac.uk

REFERENCES

Barnes, M. (2000). "Magical" Moments in Mathematics: Insights into the Process of Coming to Know. *For the Learning of Mathematics*, 20(1), 33-43.

Eizenberg, M. M., & Zaslavsky, O. (2003). Cooperative Problem Solving in Combinatorics: The Inter-Relations between Control Processes and Successful Solutions. *Journal of Mathematical Behavior,* 22(4), 389-403.

Fischbein, E., & Grossman, A. (1997). Schemata and Intuitions in Combinatorial Reasoning. *Educational Studies in Mathematics*, 34(1), 27-47.

Inglis, M., & Simpson, A. (in press). Mathematicians and the Selection Task. Proceedings of the 28th International Group for the Psychology of Mathematics Education, Bergen.

Mason, J. H., Burton, L., & Stacey, K. (1982). *Thinking Mathematically*. Avon: Addison-Wesley.

Raman, M. (2003). Key Ideas: What Are They and How Can They Help Us Understand How People View Proof? *Educational Studies in Mathematics*, 52(3), 319-325.

Simon, M. A. (1996). Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing. *Educational Studies in Mathematics*, 30, 197-210.

Skemp, R. R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, 77, 20-26.

Thurston, W. P. (1995). On Proof and Progress in Mathematics. *For the Learning of Mathematics,* 15(1), 29-37.